

LINEAR SUPERPOSITION OF MINIMAL SURFACES: GENERALIZED HELICOIDS AND MINIMAL CONES

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ABSTRACT. Observing a superposition principle for independent ∞ -harmonic functions, a family of new minimal hypersurfaces in Euclidean space is found, and that (linear combinations of) higher dimensional helicoids induce new algebraic minimal cones of arbitrarily high degree, generalizing Lawson's minimal cubic cone $y_2(x_1^2 - y_1^2) - x_2(2x_1y_1) = 0$ in \mathbb{R}^4 , and also including a family of Tkachev's minimal cubic cones.

Examples of algebraic minimal cones include those constructed by E. Cartan [2], W. Hsiang [7], H. Lawson [8], D. Ferus, H. Karcher, H. F. Münzner [4], V. Tkachev [11, 12], and G. Linardopoulos, T. Turgut, et al [6].

In 2013, generalizing the classical helicoid $z = \arctan \frac{y}{x}$ which is foliated by straight lines in \mathbb{R}^3 , new minimal hypersurfaces in \mathbb{R}^{2n+1} were constructed by J. Choe and the author [3] that can be thought of as arising by a multi-screw motion from the Clifford cone in \mathbb{R}^{2n} .

In this article, apart from pointing out an important effective linear superposition principle for a subclass of minimal surfaces, a multitude of new algebraic minimal cones is constructed, induced from linear combinations of generalized/higher dimensional helicoids.

Example 1 (CH-helicoid in \mathbb{R}^{2n+1} , [3, Theorem 2]). Let $\lambda \in \mathbb{R}$. Sweeping out the Clifford cone \mathbb{C}^{2n-1} in \mathbb{R}^{2n} given by

$$\mathbb{C}^{2n-1} = \left\{ \begin{bmatrix} p_1 \\ q_1 \\ \vdots \\ p_n \\ q_n \end{bmatrix} \in \mathbb{R}^{2n} \mid p_1^2 + \cdots + p_n^2 = q_1^2 + \cdots + q_n^2 \right\}$$

yields the CH-helicoid \mathcal{H}_λ in \mathbb{R}^{2n+1} given by

$$\mathcal{H}_\lambda = \left\{ \begin{bmatrix} x_1 \\ y_1 \\ \vdots \\ x_n \\ y_n \\ z \end{bmatrix} = \begin{bmatrix} p_1 \cos \Theta - q_1 \sin \Theta \\ q_1 \cos \Theta + p_1 \sin \Theta \\ \vdots \\ p_n \cos \Theta - q_n \sin \Theta \\ q_n \cos \Theta + p_n \sin \Theta \\ \lambda \Theta \end{bmatrix} \in \mathbb{R}^{2n+1} \mid \Theta \in \mathbb{R}, \begin{bmatrix} p_1 \\ q_1 \\ \vdots \\ p_n \\ q_n \end{bmatrix} \in \mathbb{C}^{2n-1} \right\}.$$

Keywords: Algebraic minimal cone, ∞ -harmonic functions, superposition principle.

One can check that $\mathcal{H}_{\lambda \neq 0}$ is congruent to $\lambda \mathcal{H}_1$. As observed in [9, Remark 5], one finds that the minimal variety \mathcal{H}_1 can be viewed as the multi-valued graph in $\mathbb{R}^{2n+1} = \mathbb{C}^n \times \mathbb{R}$

$$z = \arg \left(\sqrt{(x_1 + iy_1)^2 + \cdots + (x_n + iy_n)^2} \right).$$

Lemma 1 (∞ -harmonicity of the height function of the CH-helicoid). *The CH-function in $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ given by*

$$F = \mathbf{F}_{\mathbb{R}^{2n}}(X, Y) = \frac{1}{2} \arctan \left(\frac{V}{U} \right), \quad \text{where} \quad (U, V) = \left(\|X\|_{\mathbb{R}^n}^2 - \|Y\|_{\mathbb{R}^n}^2, 2\langle X, Y \rangle_{\mathbb{R}^n} \right),$$

is ∞ -harmonic, in the sense that it solves the so called infinity Laplace equation

$$0 = \Delta_{\infty} F := \left\langle \nabla F, \nabla \left(\frac{1}{2} \|\nabla F\|_{\mathbb{R}^{2n}}^2 \right) \right\rangle_{\mathbb{R}^{2n}}.$$

Proof. Straightforward computation. \square

We present an alternative proof of the minimality of the CH-helicoid, and shall see that the analytic proof motivates Theorem 1.

Lemma 2 (Minimality of the CH-helicoid). *The multi-valued graph*

$$z = \arg \left(\sqrt{(x_1 + iy_1)^2 + \cdots + (x_n + iy_n)^2} \right)$$

has zero mean curvature in Euclidean space $\mathbb{R}^{2n+1} = \mathbb{C}^n \times \mathbb{R}$.

Proof. One needs to check that the induced height function F solves the minimal hypersurface equation

$$0 = \nabla \cdot \left(\frac{\nabla F}{\sqrt{1 + \|\nabla F\|^2}} \right) = \frac{(1 + \|\nabla F\|^2) \Delta F - \Delta_{\infty} F}{(1 + \|\nabla F\|^2)^{\frac{3}{2}}}.$$

It is straightforward to check $\Delta F = 0$ (and $\Delta_{\infty} F = 0$, by Lemma 1) \square

Remark 1. In [1, Theorem 2], G. Aronsson showed that if a two-variable harmonic function $F(x, y)$ is also ∞ -harmonic, then $F(x, y) = ax + by + c$ or $F(x, y) = d \arctan \left(\frac{y - y_0}{x - x_0} \right) + e$ for some real constants a, b, c, d, e, x_0, y_0 . See also the classification result due to W. C. Graustein [5]. Both references include interesting hydrodynamic interpretations.

Remark 2. The CH-function is p -harmonic as it solves the p -Laplace equation

$$0 = \Delta_p F := \nabla \cdot \left(\|\nabla F\|^{p-2} \nabla F \right) = \|\nabla F\|^{p-4} \left[(p-2) \Delta_{\infty} F + \|\nabla F\|^2 \Delta F \right].$$

Lemma 3 (Superposition principle for ∞ -Laplacian operator). *Let $\mathbf{F}_A(u_1, \dots, u_A)$ and $\mathbf{F}_B(v_1, \dots, v_B)$ be two independent C^2 real valued functions. For real constants μ_A and μ_B , we associate the combination $\mathbf{F}(u_1, \dots, u_A, v_1, \dots, v_B) := \mu_A^{\frac{1}{3}} \mathbf{F}_A + \mu_B^{\frac{1}{3}} \mathbf{F}_B$. Then,*

$$\Delta_{\infty, \mathbb{R}^{A+B}} \mathbf{F} = \mu_A \Delta_{\infty, \mathbb{R}^A} \mathbf{F}_A + \mu_B \Delta_{\infty, \mathbb{R}^B} \mathbf{F}_B.$$

Theorem 1 (Generalized helicoids in odd dimensional Euclidean space). *The multi-valued graph of an arbitrary finite number of linear combination of independent CH height functions (introduced in Lemma 1) becomes a minimal hypersurface.*

Proof. From Lemma 1, 2, and 3, one sees that a linear combination of independent CH functions should be both harmonic and ∞ -harmonic. \square

Example 2 (Superposition principle for generalized helicoids). Let $\mu_A, \mu_B, \mu_C \in \mathbb{R}$ be constants. By Theorem 1, the multi-valued graph

$$z = \mu_A \arg \left(\sqrt{(x_1 + iy_1)^2 + \cdots + (x_4 + iy_4)^2} \right) + \mu_B \arg(x_5 + iy_5) + \mu_C \arg(x_6 + iy_6).$$

is minimal in $\mathbb{R}^{13} = \mathbb{C}^4 \times \mathbb{C} \times \mathbb{C} \times \mathbb{R}$. It is invariant under the multi-screw motion

$$(\zeta_1, \dots, \zeta_4, \zeta_5, \zeta_6, z) \rightarrow (e^{it_A} \zeta_1, \dots, e^{it_A} \zeta_4, e^{it_B} \zeta_5, e^{it_C} \zeta_6, z + \mu_A t_A + \mu_B t_B + \mu_C t_C),$$

where $\zeta_k = x_k + iy_k$ for $k \in \{1, \dots, 6\}$.

Theorem 2 (n -variable harmonic and ∞ -harmonic functions induce minimal hypersurfaces in \mathbb{R}^{n+1} and \mathbb{R}^{n+2}). Let $\mathbf{f}(z_1, \dots, z_n)$ is a C^2 -function satisfying

$$\Delta \mathbf{f} = 0 \quad \text{and} \quad \Delta_\infty \mathbf{f} = 0.$$

- (1) The graph $z_{n+1} = \mathbf{f}(z_1, \dots, z_n)$ is a minimal hypersurface in \mathbb{R}^{n+1} .
- (2) The graph $z_{n+1} = z_0 \tan \mathbf{f}(z_1, \dots, z_n)$ is a minimal hypersurface in \mathbb{R}^{n+2} .

Proof. The proofs for the two items use the same idea. First verify the item (1) using the superposition principle. The idea is to view the graph $z_{n+1} = \mathbf{f}(z_1, \dots, z_n)$ as a level set of an $(n+1)$ -variable function $\mathbf{U}(z_1, \dots, z_n, z_{n+1}) := -z_{n+1} + \mathbf{f}(z_1, \dots, z_n)$:

$$\left\{ (z_1, \dots, z_n, z_{n+1}) \in \mathbb{R}^{n+1} \mid 0 = \mathbf{U}(z_1, \dots, z_n, z_{n+1}) \right\}.$$

This level set is a minimal submanifold if and only if the function \mathbf{U} satisfies

$$0 = \sum_{k=1}^{n+1} \frac{\partial}{\partial z_k} \left(\frac{\frac{\partial \mathbf{U}}{\partial z_k}}{\|\nabla \mathbf{U}\|} \right), \quad \text{or equivalently,} \quad 0 = \|\nabla \mathbf{U}\|^2 \Delta \mathbf{U} - \Delta_\infty \mathbf{U}.$$

Since both $-z_{n+1}$ and $\mathbf{f}(z_1, \dots, z_n)$ are harmonic and ∞ -harmonic, by the superposition principle, its sum \mathbf{U} is harmonic and ∞ -harmonic. This guarantees the desired equality $0 = \|\nabla \mathbf{U}\|^2 \Delta \mathbf{U} - \Delta_\infty \mathbf{U}$. For the proof of item (2), introduce the splitting

$$\mathbf{V}(z_0, z_1, \dots, z_n, z_{n+1}) := -\arctan\left(\frac{z_{n+1}}{z_0}\right) + \mathbf{f}(z_1, \dots, z_n).$$

Since both $-\arctan\left(\frac{z_{n+1}}{z_0}\right)$ and $\mathbf{f}(z_1, \dots, z_n)$ are harmonic and ∞ -harmonic, by the superposition principle, its sum \mathbf{V} is also harmonic and ∞ -harmonic. Thus, the zero set $\left\{ (z_0, z_1, \dots, z_n, z_{n+1}) \in \mathbb{R}^{n+2} \mid 0 = \mathbf{V}(z_0, z_1, \dots, z_n, z_{n+1}) \right\}$ is minimal in \mathbb{R}^{n+2} . \square

Remark 3. The same argument of the proof of Theorem 2 shows that whenever two C^2 -functions $\mathbf{f}(z_1, \dots, z_n)$ and $\mathbf{g}(w_1, \dots, w_m)$ are harmonic and ∞ -harmonic, the level set

$$\left\{ (z_1, \dots, z_n, w_1, \dots, w_m) \in \mathbb{R}^{n+m} \mid 0 = \mathbf{f}(z_1, \dots, z_n) + \mathbf{g}(w_1, \dots, w_m) \right\}.$$

defines a (possibly singular) minimal submanifold in Euclidean space \mathbb{R}^{n+m} .

Example 3 (Helicoids in \mathbb{R}^3 and Clifford's quadratic cone in \mathbb{R}^4). The height function $\mathbf{f}(x_1, y_1) = \arctan\left(\frac{y_1}{x_1}\right)$ of the helicoid $z = \arctan\left(\frac{y_1}{x_1}\right)$ in \mathbb{R}^3 is both harmonic and ∞ -harmonic. Then, item (2) of Theorem 2 guarantees that the graph Σ given by $y_2 = x_2 \tan \mathbf{f}(x_1, y_1)$ becomes a minimal hypersurface in \mathbb{R}^4 . The hypersurface Σ is the quadratic cone $y_2 x_1 = x_2 y_1$. In fact, applying the $\frac{\pi}{4}$ -rotations in $x_1 y_2$ -plane and $x_1 y_2$ -plane, one finds that Σ is congruent to the hypersurface $y_2^2 - x_1^2 = y_1^2 - x_2^2$. The 3-fold Σ is the cone over Clifford's minimal torus $x_1^2 + y_1^2 = x_2^2 + y_2^2 = \frac{1}{2}$ in the sphere $\mathbb{S}^3 \subset \mathbb{R}^4$.

Example 4 (Lawson's algebraic minimal cones [8] of degree $N + 1$ in \mathbb{R}^4). Let $N \geq 1$ be an integer. Apply the dilation $(x_1, y_1, z_1) \rightarrow (\frac{x_1}{N}, \frac{y_1}{N}, \frac{z_1}{N})$ to the minimal surface $z = \arctan(\frac{y_1}{x_1})$ in \mathbb{R}^3 to obtain the helicoid $z = \mathbf{f}_N(x_1, y_1) = N \arctan(\frac{y_1}{x_1})$. Since the height function $\mathbf{f}_N(x_1, y_1) = N \arctan(\frac{y_1}{x_1})$ is both harmonic and ∞ -harmonic, introducing the rational polynomial $\mathbf{q}_N(t) = \tan(N \arctan t) \in \mathbb{Q}[t]$, the item (2) of Theorem 2 guarantees that the graph Σ_N given by $y_2 = x_2 \mathbf{q}(\frac{y_1}{x_1})$ is minimal in \mathbb{R}^4 . It is easy to check that Σ_N becomes a Lawson's algebraic minimal cone of degree $N + 1$ in \mathbb{R}^4 . As noticed in [8, Theorem 3 and Proposition 7.2], the 3-fold Σ_N becomes the cone over a ruled minimal surface in the sphere $\mathbb{S}^3 \subset \mathbb{R}^4$. In the case when $N = 2$, one has $\mathbf{q}_2(t) = \tan(2 \arctan t) = \frac{2t}{1-t^2}$. Then one finds that the minimal 3-fold Σ_2 is the cubic cone $y_2(x_1^2 - y_1^2) - x_2(2x_1y_1) = 0$ in \mathbb{R}^4 .

Example 5 (CH-helicoids in \mathbb{R}^{2n+1} and Tkachev's cubic cone in \mathbb{R}^{2n+2}). The harmonicity and ∞ -harmonicity of the height function

$$\mathbf{f}(x_1, y_1, \dots, x_n, y_n) = \arctan\left(\frac{2x_1y_1 + \dots + 2x_ny_n}{(x_1^2 - y_1^2) + \dots + (x_n^2 - y_n^2)}\right)$$

and the item (2) of Theorem 2 guarantee that the cone

$$y_{n+1} = x_{n+1} \left(\frac{2x_1y_1 + \dots + 2x_ny_n}{(x_1^2 - y_1^2) + \dots + (x_n^2 - y_n^2)} \right)$$

is minimal in \mathbb{R}^{2n+2} . It coincides with one of Tkachev's minimal cubic cones [11, Example 4] (V. Tkachev used Clifford algebras to construct a large class of minimal cubic cones; see also [10, Chapter 6] and [12]).

Example 6 (Higher degree generalization of Tkachev's cubic cone in \mathbb{R}^{2n+2}). For an integer $N \geq 1$, introduce the rational polynomial $\mathbf{q}_N(t) = \tan(N \arctan t) \in \mathbb{Q}[t]$. Set

- (1) $\mathbf{T}_1(x_1, y_1, \dots, x_n, y_n) = (x_1^2 - y_1^2) + \dots + (x_n^2 - y_n^2)$,
- (2) $\mathbf{T}_2(x_1, y_1, \dots, x_n, y_n) = 2x_1y_1 + \dots + 2x_ny_n$.

Since the function $\mathbf{f}_N(x_1, y_1, \dots, x_n, y_n) = N \arctan\left(\frac{\mathbf{T}_2(x_1, y_1, \dots, x_n, y_n)}{\mathbf{T}_1(x_1, y_1, \dots, x_n, y_n)}\right)$ is both harmonic and ∞ -harmonic, the item (2) of Theorem 2 guarantees that the algebraic cone

$$y_{n+1} = x_{n+1} \mathbf{q}_N\left(\frac{\mathbf{T}_2(x_1, y_1, \dots, x_n, y_n)}{\mathbf{T}_1(x_1, y_1, \dots, x_n, y_n)}\right)$$

is a minimal hypersurface in \mathbb{R}^{2n+2} . Consider the case $N = 2$. We find that $\mathbf{q}_2(t) = \tan(2 \arctan t) = \frac{2t}{1-t^2}$, and that the $(2n + 1)$ -fold Σ_2 is the minimal quintic cone in \mathbb{R}^{2n+2} :

$$y_{n+1} \left[(\mathbf{T}_1)^2 - (\mathbf{T}_2)^2 \right] - x_{n+1} (2\mathbf{T}_1\mathbf{T}_2) = 0.$$

Example 7 (Minimal quintic cones in \mathbb{R}^{4n+2}). Introduce quadratic polynomials

- (1) $\mathbf{T}_1(x_1, y_1, \dots, x_n, y_n) = (x_1^2 - y_1^2) + \dots + (x_n^2 - y_n^2)$,
- (2) $\mathbf{T}_2(x_1, y_1, \dots, x_n, y_n) = 2x_1y_1 + \dots + 2x_ny_n$,
- (3) $\mathbf{S}_1(x_{n+1}, y_{n+1}, \dots, x_{2n}, y_{2n}) = (x_{n+1}^2 - y_{n+1}^2) + \dots + (x_{2n}^2 - y_{2n}^2)$,
- (4) $\mathbf{S}_2(x_{n+1}, y_{n+1}, \dots, x_{2n}, y_{2n}) = 2x_{n+1}y_{n+1} + \dots + 2x_{2n}y_{2n}$.

From the proof of Theorem 2 and the item (2) of Theorem 2, one knows that the function

$$\mathbf{f}(x_1, y_1, \dots, x_{2n}, y_{2n}) = \theta_1 + \theta_2, \quad \text{where } \theta_1 = \arctan\left(\frac{\mathbf{T}_2}{\mathbf{T}_1}\right), \text{ and } \theta_2 = \arctan\left(\frac{\mathbf{S}_2}{\mathbf{S}_1}\right)$$

is both harmonic and ∞ -harmonic, and obtain the minimal quintic cone in \mathbb{R}^{4n+2} :

$$y_{2n+1} [\mathbf{T}_1 \mathbf{S}_1 - \mathbf{T}_2 \mathbf{S}_2] - x_{2n+1} [\mathbf{T}_2 \mathbf{S}_1 + \mathbf{T}_1 \mathbf{S}_2] = 0.$$

Example 8 (Lawson's algebraic minimal cones in \mathbb{R}^{2n}). Let k_1, \dots, k_n be positive integers with $\gcd(k_1, \dots, k_n) = 1$. Observing that angle functions $\arctan\left(\frac{y_1}{x_1}\right), \dots, \arctan\left(\frac{y_n}{x_n}\right)$ are harmonic and ∞ -harmonic, one can associate the minimal hypersurface in \mathbb{R}^{2n} :

$$\left\{ (x_1, y_1, \dots, x_n, y_n) \in \mathbb{R}^{2n} \mid 0 = k_1 \arctan\left(\frac{y_1}{x_1}\right) + \dots + k_n \arctan\left(\frac{y_n}{x_n}\right) \right\}.$$

This level set produces the algebraic minimal cone in \mathbb{R}^{2n} :

$$0 = \mathbf{Im} \left[(x_1 + iy_1)^{k_1} \dots (x_n + iy_n)^{k_n} \right].$$

It is congruent to the example constructed by H. Lawson [8, p. 352] after suitable rotations. For instance, taking $n = 3$ and $k_1 = k_2 = k_3 = 1$, one obtains the following minimal cubic cone in \mathbb{R}^6 :

$$0 = y_1 x_2 x_3 + x_1 y_2 x_3 + x_1 x_2 y_3 - y_1 y_2 y_3.$$

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REFERENCES

- [1] G. Aronsson, *On the partial differential equation $u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy} = 0$* , Ark. Mat. **7** (1968) 395–425.
- [2] E. Cartan, *Sur des familles remarquables d'hypersurfaces isoparamétriques dans les espaces sphériques*, Math. Z. **45** (1939). 335–367.
- [3] J. Choe, J. Hoppe, *Higher dimensional minimal submanifolds generalizing the catenoid and helicoid*, Tohoku Math. J. (2) **65** (2013), no. 1, 43–55.
- [4] D. Ferus, H. Karcher, H. F. Münzner, *Cliffordalgebren und neue isoparametrische Hyperachen*, Math. Z. (4) **177** (1981), 479–502.
- [5] W. C. Graustein, *Harmonic minimal surfaces*, Trans. Amer. Math. Soc. **47** (1940), 173–206.
- [6] J. Hoppe, G. Linardopoulos, O. Teoman Turgut, *New Minimal Hypersurfaces in $\mathbb{R}^{(k+1)(2k+1)}$ and $\mathbb{S}^{(2k+3)k}$* , arXiv preprint, arXiv:1602.09101 (2016).
- [7] W. Hsiang, *Remarks on closed minimal submanifolds in the standard Riemannian m -sphere*, J. Differential Geometry **1** (1967) 257–267.
- [8] H. B. Lawson, *Complete minimal surfaces in \mathbb{S}^3* , Ann. of Math. (2) **92** (1970), 335–374.
- [9] E. Lee, H. Lee, *Generalizations of the Choe-Hoppe helicoid and Clifford cones in Euclidean space*, arXiv:1410.3418 (2014), to appear in J. Geom. Anal.
- [10] N. Nadirashvili, V. Tkachev, S. Vlăduț, *Nonlinear elliptic equations and nonassociative algebras*. Mathematical Surveys and Monographs, 200. American Mathematical Society, Providence, RI, 2014. viii+240 pp. ISBN: 978-1-4704-1710-9
- [11] V. Tkachev, *Minimal cubic cones via Clifford algebras*, Complex Anal. Oper. Theory **4** (2010), no. 3, 685–700.
- [12] V. Tkachev, *On a classification of minimal cubic cones in \mathbb{R}^n* , arXiv preprint, arXiv:1009.5409 (2010).

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